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The design of the iso-velocity contour for the flow past the base of a dam with a confining bed☆

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ABSTRACT

The underground contour of a submerged rectangular dam, the angles of which are rounded off along curves of constant magnitude of the seepage flow velocity is constructed in the case when the water-permeable base is underlain by a confining bed, consisting of one horizontal and two curved sections, which are the iso velocity lines of the seepage flow. The corresponding multiparameter mixed problem in the theory of analytical functions is solved using the Riemann-Schwartz symmetry principle and a semi-inverse version of the velocity hodograph method, first proposed by Polybarinova-Kochina and Kochina. The results of numerical calculations are presented and a hydrodynamic analysis of the effect of the basic physical parameters of the model on the shape and dimensions of the underground contour of the dam and of the horizontal and curved sections of the confining bed is given.

The problem of choosing the smooth underground profiles of the foundations of hydraulic installations was considered for the first time in Ref. 1, where the so-called inverse boundary-value problem of the theory of the steady seepage of ground waters[2] was mentioned. This paper triggered the development of a whole research trend, that is, the search for the contours of hydraulic installations using some properties specified for them, and led to the appearance of a number of publications, mainly belonging to the Kazan School of Theoretical and Applied Mathematics (see Refs 3-8, for example). Using the Riemann-Schwartz symmetry principle and the semi-inverse version of the velocity hodograph method,[9-11], not only the design of the smooth underground contour of a submerged rectangular dam, with angles rounded off along the iso-velocity lines of the seepage flow, was considered[12] but the profile of the underlying curved confining bed, which is also the iso-velocity line of the seepage flow, was determined as well.

The case when a confining bed with similar properties has a more complex configuration and consists of a horizontal section and two curved sections is considered below. The limiting flow cases, associated with the degeneration of the conformal mapping parameters contained in the solution, are noted: the Polubarinova-Kochina and Kochina case when the confining bed is horizontal over the whole of its extent[1,2] or consists of two circular arcs.[12]

1. Formulation of the problem

A plane steady flow under a water impermeable underground contour of a submerged rectangular dam $ABCC_1B_1A_1$ (Fig. 1) is considered. The flow domain is bounded below by a confining bed G_1G, consisting of two curved sections G_1F_1 and FG and also, unlike the case considered earlier,[12] a horizontal section F_1EF: the magnitude of the flow velocity along them is constant, as it is for the sections of the underground dam contour BC and B_1C_1.

We introduce the complex potential of the motion $\omega = \varphi + i\psi$, where φ is the velocity potential, ψ is the stream function (the range of variation of the variable ω is shown in Fig. 2), and the complex coordinate $z = x + iy$, refered to κH and H respectively, where κ is the seepage coefficient and H is the pressure head acting on the installation. The problem consists of determining the positions of the curves BC, B_1C_1,

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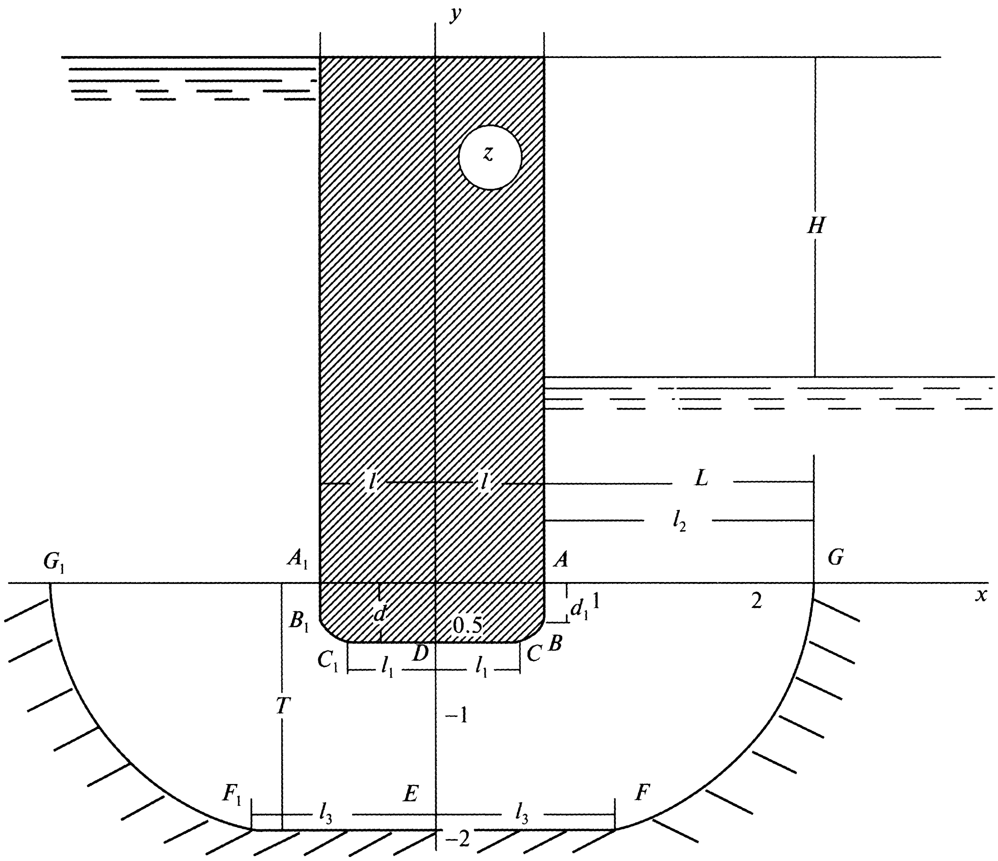
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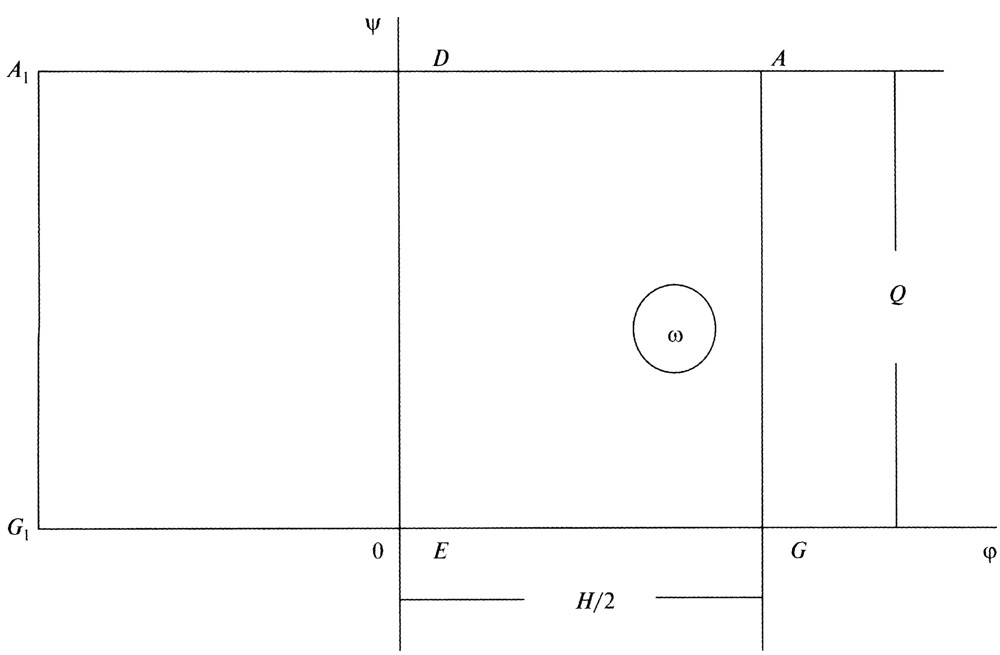
Fig. 1.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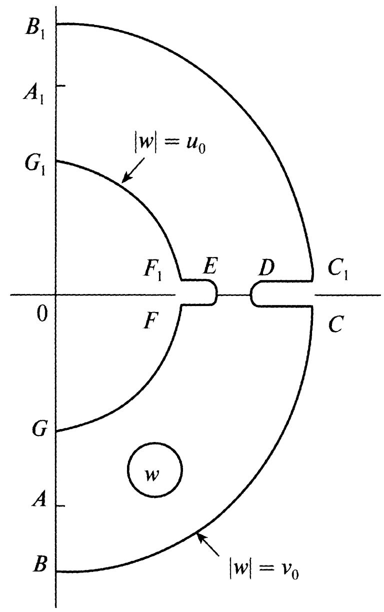

Fig. 3.

G_1F_1 and GF with the following boundary conditions

$$\begin{aligned}
&A_1G_1: y = 0, \varphi = -H/2; \quad A_1B_1: x = -l, \psi = Q\\
&C_1DC: y = -d, \psi = Q; \quad AB: x = l, \psi = Q\\
&AG: y = 0, \varphi = H/2; \quad F_1EF: y = -T, \psi = 0; \quad G_1F_1 \text{ and } FG: \psi = 0\\
&B_1C_1 \text{ and } BC: |w| = \upsilon_0; \quad G_1F_1 \text{ and } GF: |w| = u_0
\end{aligned} \tag{1.1}$$

such that the seepage flow velocity along the curved sections of the underground contour of the dam BC and B_1C_1 as well as along the horizontal section F_1EF and the curved sections G_1F_1 and GF of the confining bed have constant values υ_0 (specified) and u_0 (required) respectively ($0 \le u_0 \le \upsilon_0$).

2. Construction of the solution

Consider a domain of the complex velocity w corresponding to boundary conditions (1.1) (Fig. 3). This domain, which is a circular decagon with right angles and two cuts C_1DC and F_1EF, belongs to the class of polygons in polar meshes (see Ref. 13) which are bounded by the arcs of concentric circles and the sections of straight lines passing through the origin of coordinates.

Unlike the possibilities for solving the problem associated either with the transformation of such polygons into linear polygons with subsequent use of the Christoffel-Schwartz formula[14,15] or with the integration of the corresponding equations of the Fuchs class[16–19] [1] the Riemann-Schwartz symmetry principle[5,13] is used below, which leads to a considerable reduction in the unknown constants. The conformal mapping is carried out directly here in a closed form in terms of special functions, which is simple and convenient for subsequent practical purposes, and the unknown mapping parameters are determined simultaneously when constructing the solution.

In view of the complete symmetry in the z, ω and w planes, we will confine ourselves to considering the right-hand half of the domain of motion $ABCDEFG$ (Fig. 1) and the similar domains corresponding to it in the ω and w planes of Figs 2 and 3.

Taking account of the specific properties of polygons in polar meshes, associated with the abundance of right angles, it is convenient in the conformal mapping to take the rectangle in the τ plane[20] (Fig. 4)

$$0 < \operatorname{Re}\tau < 1/2,\ 0 < \operatorname{Im}\tau < \rho/2,\ \rho(k) = K'/K,\ K' = K(k'),\ k' = \sqrt{1-k^2} \tag{2.1}$$

as the canonical domain, where $K(k)$ is a complete elliptic integral of the first kind.[21] The function, performing the conformal mapping of this rectangle into a quadrant of the annulus of the complex velocity plane w, is expressed as

$$w(\tau) = \upsilon_0 \exp(\tau - 1/2)\pi i \tag{2.2}$$

from which the physical parameter $u_0 = \upsilon_0 \exp(-\pi\rho/2)$ is determined.

[1] Also, see: Kochina PYa, Bereslavskii EN, Kochina NN *Analytical theory of differential equations of the Fuchs class and some problems of underground hydrodynamics. Part* 1. *Preprint No.* 567. Moscow:Inst. Problem Mekhaniki Ross Akad Nauk;1996.

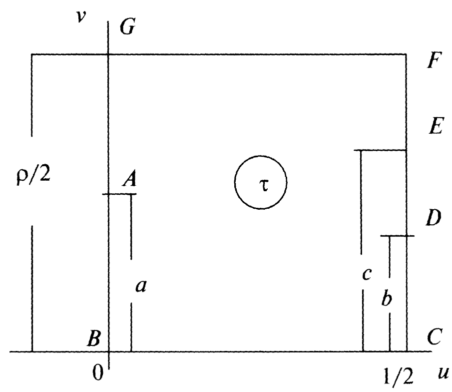

Fig. 4.

We now conformally map the rectangle of the auxiliary variable τ into the domain of the complex potential ω (Fig. 2). As a result, we obtain

$$\omega = \frac{1}{2K(k)} F\left[\arcsin\frac{\lambda}{n}\sqrt{\frac{1-n^2\mathrm{sn}^2(2K\tau,k)}{1-\lambda^2\mathrm{sn}^2(2K\tau,k)}}, m\right]$$

$$m = k\sqrt{\frac{(1-k'^2A^2B^2)(1-k'^2C^2)}{(1-k'^2B^2)(1-k'^2A^2C^2)}}, \quad \lambda = \sqrt{1-k'^2B^2}, \quad n = \sqrt{1-k'^2C^2},$$

$$A = \mathrm{sn}(2Ka,k'), \quad B = \mathrm{sn}(2Kb,k'), \quad C = \mathrm{sn}(2Kc,k') \tag{2.3}$$

Here, $F(\varphi, m)$ is an elliptic integral of the first kind,[21] and $\mathrm{sn}(\varphi, k)$, $\mathrm{cn}(\varphi, k)$ and $\mathrm{dn}(\varphi, k)$ are Jacobi elliptic functions.

At the same time, the relation

$$\rho(m) = \frac{K'(m)}{K(m)} = \frac{2Q}{H} \tag{2.4}$$

connecting the physical parameters Q and H must be satisfied.

To solve the problem, we use the first version of the velocity hodograph method (Ref. 9, pp. 250,251; Ref. 10, p. 60; Ref 11, pp. 603-606). Taking account of relations (2.2) and (2.3) and proceeding in a similar way to that described earlier,[22,23] we arrive at the relations

$$\frac{d\omega}{d\tau} = -\frac{M\mathrm{sn}(2K\tau,k)\mathrm{cn}(2K\tau,k)\mathrm{dn}(2K\tau,k)}{\Delta(\tau)}$$

$$\frac{dz}{d\tau} = -\frac{M\mathrm{sn}(2K\tau,k)\mathrm{cn}(2K\tau,k)\mathrm{dn}(2K\tau,k)\exp((1/2-\tau)\pi i)}{\upsilon_0\Delta(\tau)}$$

$$\Delta(\tau) = \sqrt{\left[1-\lambda^2\mathrm{sn}^2(2K\tau,k)\right]\left[1-n^2\mathrm{sn}^2(2K\tau,k)\right]\left[A^2+(1-A^2)\mathrm{sn}^2(2K\tau,k)\right]} \tag{2.5}$$

where $M > 0$ is the modelling scale constant. Writing the representations (2.5) for the different sections of the boundary of the domain τ, followed by integration over the whole of the contour of the auxiliary variable, we obtain the following expressions:

for the main geometric and seepage characteristics

$$\frac{M}{\upsilon_0}\int_0^{1/2} X_{BC}\mathrm{dt} = \Delta l, \quad \frac{M}{\upsilon_0}\int_0^{1/2} Y_{BC}\mathrm{dt} = \Delta d, \quad M\left(\int_0^{\rho/2}\Phi_{EF}\mathrm{dt} + \int_0^{1/2}\Phi_{FG}\mathrm{dt}\right) = \frac{H}{2}, \quad \frac{M}{u_0}\int_0^{1/2} Y_{FG}\mathrm{dt} = T \tag{2.6}$$

for the coordinates of the points of the contour of the apron BC

$$x_{BC}(t) = l - \frac{M}{\upsilon_0}\int_0^t X_{BC}\mathrm{dt}, \quad y_{BC}(t) = -d_1 - \frac{M}{\upsilon_0}\int_0^t Y_{BC}\mathrm{dt}, \quad 0 \le t \le 1/2 \tag{2.7}$$

and for the coordinates of the curvilinear part of the confining bed FG

$$x_{FG}(t) = L - \frac{M}{u_0}\int_0^t X_{FG}\mathrm{dt}, \quad y_{FG}(t) = -\frac{M}{u_0}\int_0^t Y_{FG}\mathrm{dt}, \quad 0 \le t \le 1/2 \tag{2.8}$$

Here,

$$\Delta l = l - l_1,\ \Delta d = d - d_1,\ L = l + l_2,\ X_{FG} = \sin\ \pi t\ \Phi_{FG},\ Y_{FG} = \cos\ \pi t\ \Phi_{FG}$$

$$X_{BC} = \sin\pi t\frac{\text{sn}\ (2Kt,k)\text{cn}(2Kt,k)\text{dn}(2Kt,k)}{\Delta(t)},\ Y_{BC} = \cos\pi t\frac{\text{sn}(2Kt,k)\text{cn}(2Kt,k)\text{dn}(2Kt,k)}{\Delta(t)}$$

$$\Phi_{EF} = \frac{\text{sn}(2Kt,k')\text{cn}(2Kt,k')}{\Delta_1(t,1,k',k)},$$

$$\Phi_{FG} = k\frac{\text{cn}(2Kt,k)\text{dn}(2Kt,k)}{\Delta_2(t)},\quad \Phi_{AG} = \frac{\text{sn}(2Kt,k)\text{dn}(2Kt,k)}{\Delta_3(t,1,k)}$$

$$\Delta_1(t,q,r,k) = \sqrt{q\left[\text{sn}^2(2Kt,k) - B^2\right]\left[\text{sn}^2(2Kt,k) - C^2\right]\left[1 - r^2A^2\text{sn}^2(2Kt,k)\right]}$$

$$\Delta_2(t) = \sqrt{\left[\text{dn}^2(2Kt,k) - \lambda^2\right]\left[n^2 - k^2\text{sn}^2(2Kt,k)\right]\left[1 - A^2\text{dn}^2(2Kt,k)\right]}$$

$$\Delta_3(t,p,k) = \sqrt{p\left[1 - \lambda'^2\text{sn}^2(2Kt,k)\right]\left[1 - n'^2\text{sn}^2(2Kt,k)\right]\left[\text{sn}^2(2Kt,k) - A^2\right]}$$

Putting $t = ½$ in Eqs (2.7) and (2.8) we find the required dimensions of the underground contour of the apron and the confining bed

$$l_1 = x_{BC}(1/2),\quad d_1 = y_{BC}(1/2),\quad l_2 = \frac{M}{\upsilon_0}\int_a^{\rho/2}\Phi_{AG}\exp(\pi t)\text{d}t,\quad l_3 = L - x_{FG}(1/2), \tag{2.9}$$

The other expressions for the flow rate Q and the geometrical dimensions l_1, d_1, l_3 and T are the control of the calculation:

$$Q_{AG} = M\int_0^{\rho/2}\Psi_{AG}\text{d}t,\quad Q_{DE} = M\int_b^c\Psi_{DE}\text{d}t,\quad Q = Q_{AG} = Q_{DE}$$

$$l_1 = \frac{M}{\upsilon_0}\int_0^{в}X_{CD}\text{d}t,\quad d_1 = \frac{M}{\upsilon_0}\int_0^{a}Y_{AB}\text{d}t,\quad l_3 = \frac{M}{\upsilon_0}\int_0^{\rho/2}X_{EF}\text{d}t,\quad T = d + \frac{M}{\upsilon_0}\int_b^{\rho/2}Y_{DE}\exp(\pi t)\text{d}t \tag{2.10}$$

Here,

$$X_{CD} = \frac{\text{sn}(2Kt,k)\text{cn}(2Kt,k)\exp(\pi t)}{\Delta_1(t,1,k',k')},\ Y_{AB} = \frac{\text{sn}(2Kt,k)\text{dn}(2Kt,k)\exp(\pi t)}{\Delta_3(t,-1,k')}$$

$$\Psi_{AG} = \frac{\text{sn}(2Kt,k)\text{dn}(2Kt,k)}{\Delta_3(t,1,k)},\ \Psi_{DE} = \frac{\text{sn}(2Kt,k)\text{cn}(2Kt,k)}{\Delta_1(t,-1,k',k')}$$

$$X_{EF} = \frac{\text{sn}(2Kt,k)\text{cn}(2Kt,k)\exp(\pi t)}{\Delta_1(t,1,k',k')}$$

3. Limiting cases. The Polubarinova-Kochina and Kochina case

We will first cinsider the case[1,2] when the confining bed is horizontal along its whole extent. Then, in the plane of motion z, the points G and F merge at infinity and the rectangle of the plane of the auxiliary variable τ is transformed into the half-strip $0 < \text{Re}\tau < 1/2$, $0 < \text{Im}\tau < \infty$ (Fig. 4), since $k = 1$, $k' = 0$, $K = \pi/2$, $K' = \infty$ and, consequently, $\rho = \infty$. The solution in this limiting case is obtained from formulae (2.5)-(2.10) if we put $k = 0$ in them. It follows from formulae (2.9) and (2.10) that, at the same time, $l_2 = \infty$, $l_3 = \infty$, and expressions (2.6) for H and T can be integrated in explicit form:

$$H = \frac{2MK(k)}{\pi\sqrt{(1 - A^2B^2)(1 - C^2)}},\quad T = \frac{M}{\upsilon_0\sqrt{(1 - A^2)(1 - B^2)(1 - C^2)}}$$

$$k = \sqrt{\frac{(1 - A^2B^2)(1 - C^2)}{(1 - A^2C^2)(1 - B^2)}} \tag{3.1}$$

Formulae (3.1) are identical with known formulae (Ref. 2, p. 19, formulae (7.17) and (7.18)) if account is taken of the fact that the parameters α and β from Ref. 2 are related to the parameters used here as follows:

$$\alpha\frac{C}{B} = \beta = \frac{1}{k}\sqrt{\frac{1 - C^2}{1 - B^2}}$$

Another limiting case is obtained from the general scheme if the points F_1, F and E in the plane of flow merge, that is, when there is no horizontal impermeable section and the confining bed turns out to be curvilinear over its whole extent.[12] In this case, in the τ plane the

Table 1

parameter varied	$l_1 \cdot 10^3$	$d_1 \cdot 10^3$	$l_2 \cdot 10^3$	$l_3 \cdot 10^3$	parameter varied	$l_1 \cdot 10^3$	$d_1 \cdot 10^3$	$l_2 \cdot 10^3$	$l_3 \cdot 10^3$
$\upsilon_0 \cdot 10^3 = 85$	838	375	2038	1835	$H = 1.2$	85	76	2646	900
90	662	329	2035	1677	1.4	170	133	2410	1028
100	397	161	1640	1051	1.6	273	186	2228	1160
120	359	118	1026	1018	1.8	395	230	2072	1288
$Q = 1.4$	571	226	2426	2014	$T = 1.1$	710	55	1265	1543
1.6	587	209	2808	2439	1.3	682	101	1454	1562
1.8	597	200	3192	2846	1.7	598	202	1795	1626
2.0	601	195	3579	3245	1.9	553	246	2020	1668
$\Delta l \cdot 10^2 = 30$	298	290	2233	1509	$\Delta d \cdot 10^3 = 25$	340	588	2135	1519
37	251	495	2196	1426	30	566	230	2030	1560
44	88	676	2174	1331	35	721	67	1949	1600
50	0	735	2173	1254	40	757	0	1928	1618

parameter $c = \rho/2$, and the solution is obtained from formulae (2.5)-(2.10) if we put $C = 1$ in them. Here, it follows from formulae (2.9) and (2.10) that $l_3 = 0$.

4. Calculation scheme and analysis of the numerical results

Representations (2.5)–(2.10) contain five unknown constants A, B, C, k and M. Relation (2.4), the right-hand side of which (by virtue of the asymptotics[21] $K'/K = \pi/(2\ln(4/k'))$) cannot be specified in an arbitrary manner and lies in a certain range, serves to determine the modulus of the elliptic integrals k. The latter is determined by the critical values Q_* and H_*, which correspond to the cases when $k \approx 0$ and $k \approx 1$. Note that a similar situation has already arisen in numerical calculations.[24] The three other mapping parameters A, B and C ($0 < A < 1$, $0 < B < C \leq 1$) are determined from Eqs (2.6) for given values of Δl, Δd and T and the modelling constant M is found in advance from the third equation of (2.6), which fixes the acting pressure head H. After the unknown constants have been found, the required dimensions of the underground contour of the installation l_1 and d_1 and the quantities l_2 and l_3 are successively found using formulae (2.9), the width and lowering of the dam $l = l_1 + \Delta l$ and $d = d_1 + \Delta d$ and, finally, the coordinates of the points of the underground contour of the installation BC and the curvilinear part FG are calculated using formulae (2.7) and (2.8) respectively.

The flow pattern, calculated when

$$\upsilon_0 = 1, \quad H = 2, \quad Q = 1.14, \quad T = 1.934, \quad \Delta l = 0.308, \quad \Delta d = 0.295 \tag{4.1}$$

is shown in Fig. 1.

Results of calculations of the effect of the governing physical parameters υ_0, H, Q, T, Δl, Δd on the dimensions l_1, d_1 (and, consequently, on l and d), l_2 and l_3 are shown in Table 1, which is subdivided by the double lines into six blocks. In each of them, one of the above mentioned parameters is varied (over a permissible range) and the values of the remaining parameters are fixed in accordance with equalities (4.1). Graphs of d_1 and l_1 against the parameter T are shown in Fig. 5 and graphics of l_2 and l_3 against the parameter Q are shown in Fig. 6.

Analysis of the data in Table 1 and the graphs anables the following conclusions to be drawn.

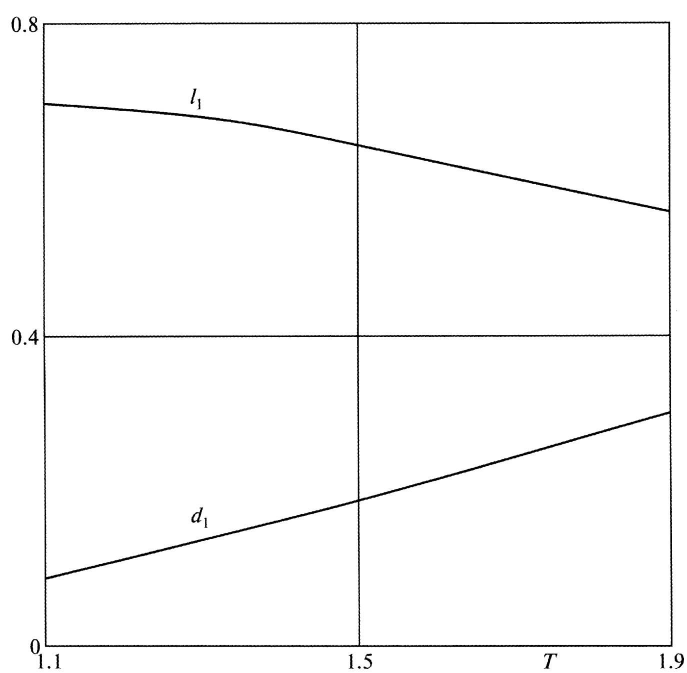

Fig. 5.

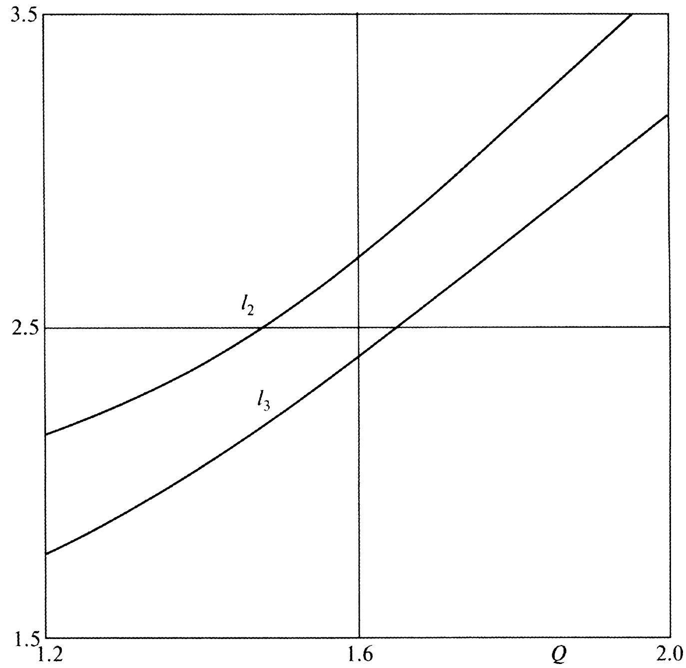

Fig. 6.

A decrease in the flow rate and an increase in the pressure head acting on the installation lead to an increase in all the dimensions of the dam as well as of the magnitude of the horizontal section of the confining bed. According to the data in the upper left block of Table 1, increasing the rate by a factor of 1.4 leads to an increase in the width l_1 and the depth d_1 by a factor of 2.3 and 3.2 respectively. At the same time, the most important influence on the width of the dam and its depth is found to be the acting pressure head: the data in the upper right block of Table 1 show that, when the parameter H increases by a factor of 1.5, the values of l_1 and d_1 increase by a factor of 4.6 and 3 respectively.

It follows from the data in the middle left block of Table 1 that an increase in the parameter Q by a factor of 1.4 leads to very insignificant changes in the sizes of l_1 and d_1 (by 5% and 16% respectively) so that the seepage flow rate has barely any effect on the dimensions of the dam. There is a noteworthy tendency for the width of the dam l_1 to increase when the seepage flow rate Q is increased and the thickness of the bed T is reduced and, also, for an increase in the depth d_1, conversely, when the parameter Q is decreased and T is increased (Fig. 5). It is clear from the data in the middle right block of Table 1 that, together with the parameter H, the thickness of the bed has a strong effect on the depth d_1, changing the latter by a factor of 4.6.

The lower blocks of Table 1 reflect a regularity which is natural from a physical point of view: a decrease (growth) in the width of the dam l and an increase (decrease) in its depth d leads to increase in the difference $\Delta l(\Delta d)$. For instance, when Δl changes by 47%, the width l_1 decreases by a factor of 3.4 and the depth d_1 increases by a factor of 2.3 and, when Δd changes by 40%, the width of the dam l_1 increases by the same factor of 2.3 while the depth d_1 now decreases by a factor of 8.8. A different behaviour of the dam dimensions l_1 and d_1 is observed when the parameters T and Δd are varied and, conversely, the same qualitative from of the dependence of these dimensions on the parameters T and Δl.; an increase in the latter parameters leads to an increase in the depth of the dam d_1 and a reduction in its width l_1. The last row of Table 1 corresponds to cases of flow past a raffet (tooth) when $l_1 = 0$, $l = \Delta l$ and an apron with a horizontal insertion where $d_1 = 0$, $d = \Delta d$.

The character of the emergence of water into the lower race l_2 and of the size of the horizontal segment of the confining bed l_3 are of special interest. According to the data in the middle blocks of Table 1 and Fig. 6, the width l_2 increases as the parameters Q and T increase and decreases as υ_0, H, Δl and Δd increase. The values of l_2 and l_3 can be very significant here. For instance, it follows from the data in the middle left block of Table 1 that $l = 0.909$, $d = 0.489$, $l_2/l = 3.9$, $l_3/l = 3.6$, $l_2/d = 7.3$, $l_3/d = 6.6$ when $Q = 2$. For all the blocks in Table 1, it is noteworthy that $l_2 > l_3$ when $T > 1.5$ and that the ratio l_2/l_3 can be close to three (see the upper right block of Table 1).

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